

Uniformly Deployable Kirigami on Arbitrary Planar Graphs

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We present an analytical framework for exploring the design space of hinged kirigami structures that deploy rigidly and uniformly. A hinged kirigami structure consists of rigid planar faces connected by hinges, and is deployable if the faces can rotate about the hinges without deformation. Such deployability depends on both the topology and the geometry, i.e., the combinatorial connectivity of the faces and their spatial embedding. Prior work studies geometric constraints for uniform deployability in restricted settings, most notably quadrilateral kirigami patterns and structures derived from 2-colorable planar graphs. However, existing analysis does not readily generalize to arbitrary kirigami structures, nor does it provide a systematic approach for constructing deployable kirigami from non-2-colorable planar graphs (e.g., graphs with non-manifold embeddings). Moreover, while geometric conditions for deployability have been partially investigated, the structure of the full deployable design space and its associated degrees of freedom remain largely unexplored. In this work, we propose a new framework that enables the derivation of hinged kirigami structures from arbitrary planar graphs, with the key feature that multiple distinct kirigami structures can be generated from the same graph. We derive geometric constraints that ensure uniform rigid deployability and analytically characterize the full design space of deployable embeddings, including its associated degrees of freedom. This characterization allows continuous navigation of the design space and provides a systematic foundation for the design of kirigami-based mechanical metamaterials.

CCS Concepts: • **Computing methodologies** → **Shape modeling**.

Additional Key Words and Phrases: Computational fabrication, kirigami, auxetic metamaterials

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1 Introduction

Kirigami extends origami by introducing cuts into flat sheets, thereby enabling relative motion between rigid regions and substantially enlarging the space of attainable deformations. This principle underlies a wide range of deployable structures and mechanical metamaterials, with applications including shape morphing and programmable curvature [Konaković et al. 2018; Choi et al. 2019; Jiang et al. 2022; Segall et al. 2025; Dudek et al. 2025], tunable material behavior [Tang et al. 2015; Tang and Yin 2017; Tang et al. 2019] and mechanical functionalities [Lee et al. 2012; Schenk and Guest 2013].

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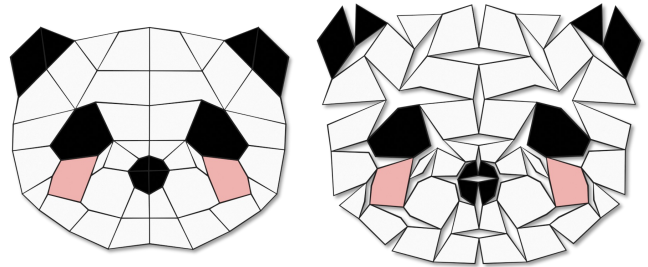


Fig. 1. An example of a non-two-colorable, aperiodic planar graph that admits a uniform deployable kirigami structure, shown before (left) and after (right) deployment, with a panda texture.

In this work, we focus on *rigid hinged* kirigami, in which cutting partitions a flat, non-stretchable sheet into rigid planar faces connected by hinges. Deformation occurs exclusively through hinge rotations, while the faces remain rigid. Within this setting, a kirigami structure is fully characterized by its *topology*, which encodes the combinatorial arrangement of faces and hinges induced by cutting, and its *geometry*, which specifies the planar embedding of vertices and faces. Together, topology and geometry define a high-dimensional design space of admissible embeddings for hinged kirigami structures. A particularly important class within this space is formed by *uniformly* deployable structures [Segall et al. 2025], where all hinges open by the same angle during deployment. Uniform deployment leads to coherent global motion and is closely related to auxetic behavior.

Uniform deployability is highly restrictive, and most kirigami patterns derived from cutting planar tilings do not admit it. Existing studies of uniform deployability have largely focused on specific pattern families. Quadrilateral kirigami patterns admit well-understood local geometric characterizations [Choi et al. 2021; Dang et al. 2021, 2022], and deployability has been analyzed for kirigami patterns derived from 2-colorable planar tilings (all interior vertices have even valency) [Segall et al. 2025]. Other approaches rely on pattern classes that are known a priori to be uniformly deployable, such as triangular [Konaković et al. 2016, 2018] and rectangular patterns [Jiang et al. 2020]. For general kirigami patterns, however, the relationship between topology, geometry and uniform deployability remains underexplored. In particular, for an arbitrary planar graph, it is unclear how to characterize the space of embeddings that, after cutting, yield a uniformly deployable kirigami structure, or how many degrees of freedom such embeddings may possess.

This paper develops a general framework for analyzing uniform deployability in kirigami derived from arbitrary *planar 3-connected* graphs, following the same assumption as in [Tutte 1963], which excludes only degenerate cases (such as interior faces with a single adjacent face). We use face orientations to encode the topology

of the resulting kirigami structures, where the orientations prescribe a cutting pattern that transforms a planar embedding into a hinged kirigami structure. We show that the deployability of such a structure is governed by the geometry of *holes* that emerge after cutting. Each hole induces a geometric constraint on the embedding of the original uncut structure. We further show that these geometric constraints admit a linear formulation. Solving the corresponding linear system yields embeddings that produce uniformly deployable kirigami structures, which we refer to as *Tutte auxetic embeddings*. The complete set of such embeddings characterizes the design space of uniformly deployable kirigami structures associated with a given planar graph and face orientation assignment. This formulation enables systematic design-space exploration and pattern geometry optimization, extending deployability analysis beyond previously studied pattern classes and providing a unified geometric perspective on rigid kirigami deployment (see Fig. 2).

2 Related work

2.1 Design space of deployable kirigami structures

A kirigami structure can be viewed as a rigid-face mechanism, whose degrees of freedom are determined by the graph combinatorics (topology) and the geometry of its faces. Existing studies explore this design space along limited and often decoupled dimensions. Some works fix the face geometry and analyze variations in topology: for example, Chen et al. [2020] study square-based kirigami patterns with different link arrangements and derive degrees of freedom for specific connectivities. Other works fix the topology and attempt to parameterize the design space geometrically. Works by Rafsanjani and Pasini [2016] and Chen et al. [2021] parameterize two families of kirigami patterns based on cut-edge lengths and relative angles. Their deployability is realizable under material stretchability assumptions, which limits applicability to rigid-faced kirigami. Shan et al. [2015] propose geometric parameterizations based on cut-edge lengths for three specific patterns, but their method lacks a theoretical justification for why these parameters guarantee deployability.

Another line of work focuses on identifying *geometric constraints* on embeddings that enable deployability for kirigami structures with fixed topology. For specific pattern families, deployability has been well studied. Equilateral triangular patterns deploy conformally [Konaković et al. 2016, 2018], while rectangular quadrilateral patterns deploy with constant mean stretch [Jiang et al. 2022]. The deployability of quadrilateral patterns with *non-rectangular* geometry has been formalized more generally. Choi et al. [2021] establish deployability conditions for genus-zero tessellations, showing that parallelogram-shaped voids enable continuous rigid rotation of faces. These conditions are subsequently extended to spherical embeddings [Dang et al. 2022] and planar tessellations with higher genus [Dang et al. 2021]. Collectively, these works characterize the geometric conditions under which a fixed quadrilateral combinatorial pattern admits continuous deployment. More recently, Segall et al. [2025] generalize deployability conditions to arbitrary *2-colorable* planar graphs, characterized by even-valency interior vertices, of which quadrilateral patterns are a special case. Their framework provides a method to verify whether a given embedding

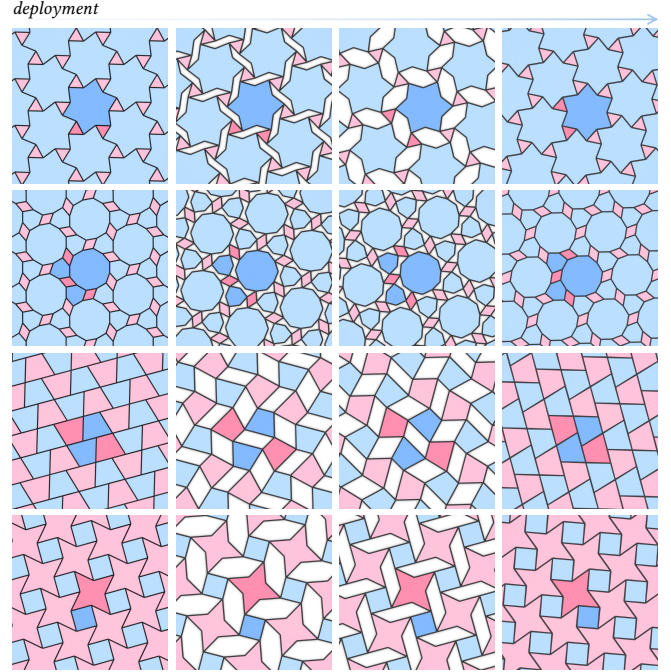


Fig. 2. Each row shows a Tutte auxetic embedding with a distinct structure, featuring two *gap-free* configurations (leftmost and rightmost columns). These patterns exhibit conformal behavior during deployment and can serve as *isotropic* kirigami structures.

is deployable; however, it does not prescribe how to modify non-deployable embeddings, nor does it characterize the full space of deployable configurations when the conditions are satisfied.

Despite the advances above, a general theoretical framework for describing the full design space of deployable kirigami structures remains elusive. Classical rigidity theory [on Rigidity Theory et al. 2002] and bar-and-joint frameworks [Mitschke et al. 2013] offer tools for analyzing configuration spaces of graphs with edge-length constraints, but they are typically restricted to *infinitesimal* mechanisms or allow deformation of faces, limiting their applicability to rigid-faced kirigami undergoing finite deployment. In this work, we provide an analytic characterization of the deployable design space for *arbitrary* 3-connected planar graphs, independent of colorability or vertex valency constraints. We introduce a formalization that parameterizes the set of all embeddings that admit uniform deployability and explicitly characterizes its degrees of freedom, enabling systematic exploration of diverse deployable kirigami structures (see Fig. 2), rather than focusing on isolated constructions or single feasible embeddings.

2.2 Kirigami-based inverse design

Computational fabrication has developed a broad class of inverse design techniques for realizing programmable curvatures across diverse materials, including flat sheets such as paper and metal [Demaine and Tachi 2017; Callens and Zadpoor 2018; Stein et al. 2018; Jiang et al. 2020; Ion et al. 2020; Binninger et al. 2021; Zhao et al. 2022; Narumi et al. 2023; He et al. 2024], yarns used in knitting or

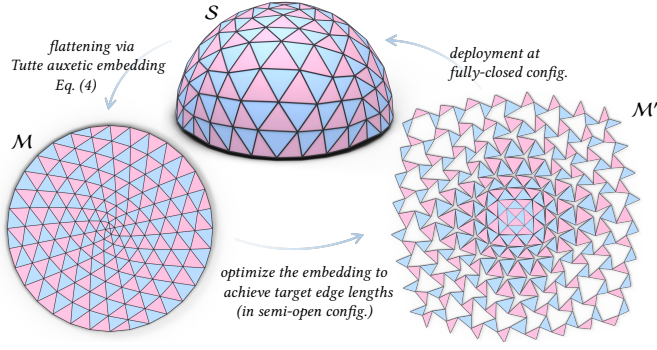


Fig. 3. The disk-topology hemisphere shape S can be regarded as a planar graph embedded in 3D. We compute its Tutte auxetic embedding M via Eq. (4), fixing the boundary vertices to a circle. This allows optimizing the kirigami structure M' to reproduce the topology and approximate the geometry of the original mesh after deployment.

crocheting [Narayanan et al. 2018; Wu et al. 2019; Edelstein et al. 2022], fabrics [Zhang et al. 2019; Jourdan et al. 2020; Segall et al. 2024; Chang et al. 2025], inflatable materials [Skouras et al. 2014; Jin et al. 2020; Panetta et al. 2021; Ren et al. 2024], stretchable materials [Jourdan et al. 2022; Guseinov et al. 2017; Shimoda et al. 2023] and auxetic materials [Konaković et al. 2016, 2018; Chen et al. 2021; Jiang et al. 2022]. Within this broader landscape, kirigami-based inverse design, often realized through auxetic cut-and-hinge patterns, focuses on prescribing planar geometries whose deployment induces target curvatures or shapes under geometric constraints. Existing kirigami inverse design approaches primarily target specific deployment regimes, including fully closed to semi-open configurations [Konaković et al. 2016], semi-open to fully open configurations [Konaković et al. 2018], semi-closed to fully open configurations [Jiang et al. 2022] and more recently, fully closed to fully closed configurations, in which curvature arises from changes in topological connectivity rather than geometric stretching [Choi et al. 2019; Dudte et al. 2023; Segall et al. 2025]. In this work, we use the new patterns generated by our framework for inverse design in different deployment regimes. In particular, our framework supports inverse design that exactly reproduces the triangulation of an input disk-topology shape, while existing methods usually require remeshing the 3D shape (see Fig. 3).

3 Background: notations & definitions

Planar graph and face orientations. Consider a planar 3-connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. By Tutte's Theorem [Tutte 1963], there exists a 2D straight-line embedding X of $\mathcal{G}$ in which no edges intersect. This embedding induces a set of faces $\mathcal{F}$, defined as the bounded regions of the plane determined by the embedding. We denote by $M = (X, \mathcal{F})$ the resulting 2D disk-topology mesh. A face-orientation assignment is defined as a function $\sigma : \mathcal{F} \rightarrow \{-1, 1\}$ that assigns clockwise ($\sigma(f) = 1$) or counterclockwise ($\sigma(f) = -1$) orientations to each face in the planar graph (see Fig. 5 a). The orientation induces a *cyclic ordering* of the edges along the boundary of each face (see Fig. 4 a,b). Accordingly, every edge incident to a face gives rise to a *directed* half-edge whose

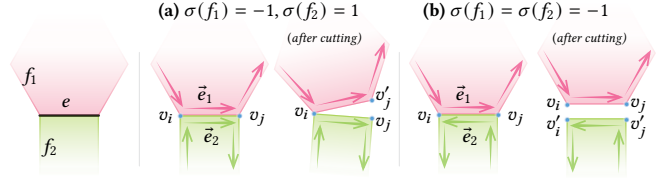


Fig. 4. **Cutting operation.** Consider an interior edge e shared by two faces f_1 and f_2 . (a) **Hinge cut**: the two faces have different orientations; the associated half edges $\vec{e}_1, \vec{e}_2$ of e have the same direction ($v_i \rightarrow v_j$). After cutting, the target vertex v_j is duplicated while the source vertex v_i forms a hinge. (b) **Split cut**: the two faces have the same orientation. After cutting, both endpoints are duplicated, fully separating the faces.

direction is consistent with the orientation of that face. We assume a planar graph and its face-orientation assignment are given as input, where the face-orientations can be either manually specified by users or automatically generated (see Sec. 4.1). Face orientations determine how edges behave after cutting.

Cutting operations. We define cutting operations along interior edges of the planar graph M . Let $e = f_1 \cap f_2$ be an interior edge shared by two faces f_1 and f_2 , and let $\vec{e}_1 \in f_1$ and $\vec{e}_2 \in f_2$ denote the associated half-edges of e . Depending on the relative orientations of f_1 and f_2 , two cases arise (see Fig. 4):

- **Hinge cuts:** If $\sigma(f_1) \neq \sigma(f_2)$, the induced half-edges along e have the same direction, say $\vec{e}_1 = \vec{e}_2 = (v_i \rightarrow v_j)$. In this case, e is treated as a *directed* edge, denoted as $\vec{e}$, and inherits this direction. A *hinge cut* is applied: the target vertex v_j is duplicated while the source vertex v_i is preserved as a hinge. After cutting, the two copies of $\vec{e}$ share the hinge vertex v_i and may rotate relative to each other. The angle between them in a deployed configuration is called the *opening angle* of $\vec{e}$ and is denoted by $\theta_{\vec{e}}$.
- **Split cuts:** If $\sigma(f_1) = \sigma(f_2)$, the induced half-edges along e have opposite directions. In this case, e is treated as an *undirected* edge and a *split cut* is applied: both endpoints of e are duplicated, fully separating the two incident faces along the edge. No hinge is formed along a split-cut edge.

Boundary edges are incident to exactly one face and are not affected by cutting. For notational convenience, we denote by $\mathcal{E}_{\text{border}}$ the set of boundary edges, by $\mathcal{E}_{\text{hinge}}$ the set of interior edges undergoing hinge cuts, and by $\mathcal{E}_{\text{split}}$ the set of interior edges undergoing split cuts. See Fig. 5 (b) for the color-coded edge classifications.

Kirigami structure. Consider the 2D embedding $M = (X, \mathcal{F})$ with predefined face orientations σ . Applying the cutting operations to all interior edges, i.e., hinge cuts for edges in $\mathcal{E}_{\text{hinge}}$ and split cuts for edges in $\mathcal{E}_{\text{split}}$, produces a new, non-manifold mesh $M' = (X', \mathcal{F}')$ (see Fig. 5). The vertex set X' consists of the original vertices together with their duplicates induced by cutting, while the face set $\mathcal{F}'$ is in one-to-one correspondence with the original face set $\mathcal{F}$ with updated vertex indices reflecting the cutting operations. We refer to M' as a kirigami structure, as all interior edges are opened, and adjacent faces in M' are connected only through hinge vertices induced by hinge cuts.

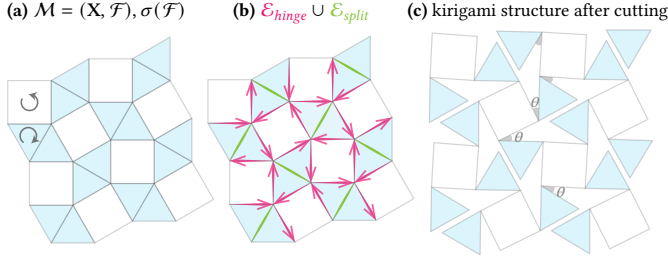


Fig. 5. **Definitions.** (a) **Face orientations** assigned to the embedded planar graph M (clockwise/counterclockwise faces are colored in blue/white). (b) **Interior edges** are partitioned based on adjacent face orientations. $\mathcal{E}_{hinge}$ includes hinge-cut edges between faces with opposite orientations and are visualized as directed edges following their half-edge directions. $\mathcal{E}_{split}$ includes split-cut edges between faces with the same orientation and are visualized as undirected edges. (c) **Kirigami structure**, the non-manifold mesh M' obtained by applying cutting operations to all interior edges of M . Highlighted gray angles indicate uniform opening angles.

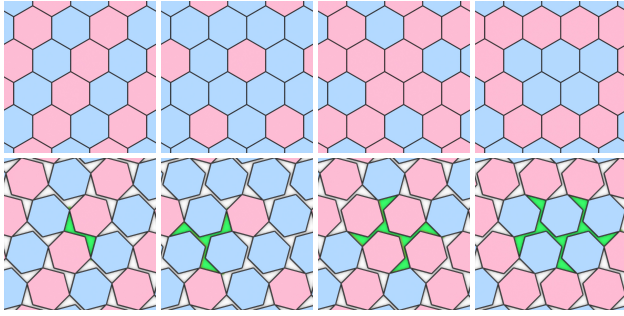


Fig. 6. **Top:** four kirigami patterns in the same geometry (hexagonal tiling) but different topologies resulting from different face-orientation assignments. **Bottom:** their deployed configurations at 30° opening angle, with one hole highlighted in green to indicate topological differences.

Uniform deployability. Following Segall et al. [2025], a kirigami structure is called *deployable* if its faces can rotate rigidly by non-trivial angles about hinges without colliding with neighboring faces. If, in addition, all hinge-cut edges open by the same angle, i.e., $\theta_e = \theta, \forall e \in \mathcal{E}_{hinge}$, the structure is called *uniformly deployable*. Note that there can be multiple sets of hinge angles that allow for a rigid deployment. Uniform deployability is the ability of a structure to deploy rigidly when all hinge angles are equal. An example of a uniformly deployable kirigami structure is shown in Fig. 5 (c), where all opening angles (some highlighted in gray) are equal. Furthermore, an embedded planar graph M is *uniformly deployable* if there exists a face-orientation assignment such that the resulting kirigami structure, after cutting, is fully connected and uniformly deployable. We call a planar graph *auxetic* if it admits a uniformly deployable embedding.

4 Method

In this paper, we address the following question:

Problem 4.1. Given an arbitrary planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ with face-orientation assignment σ , does there exist an embedding X such

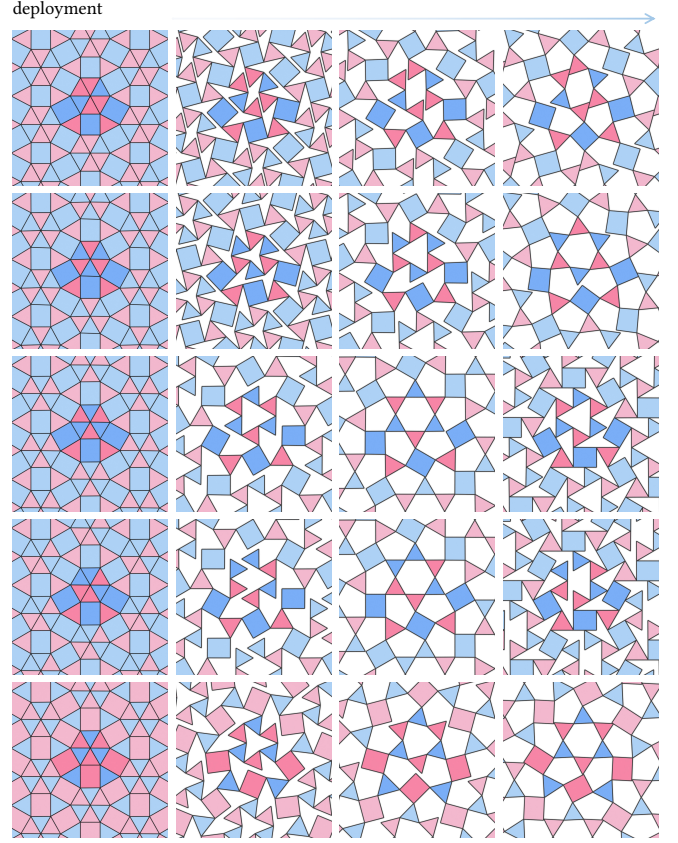


Fig. 7. Each row shows a distinct kirigami pattern derived from the same planar graph but with different face orientations, along with its deployment process up to the maximum opening angle. Note that the face geometries are adapted to ensure the deployability constraint.

that the resulting kirigami structure $M' = (X', \mathcal{F}')$, obtained by cutting the mesh $M = (X, \mathcal{F})$, is uniformly deployable? If so, what is the dimension and structure of the space of such embeddings?

The face-orientation assignment σ plays a central role in determining the *topology* of the resulting kirigami structure, since the cutting operations are induced by the relative orientation of adjacent faces. Importantly, σ is *not* unique: the same planar graph can admit multiple face-orientation assignments, leading to deployable kirigami patterns with distinct topologies, as shown in Figures 6 and 7. A key concept in our analysis is the relationship between face orientations and the formation of *holes* in the resulting kirigami structure. Holes are not present in the original mesh M ; they arise topologically after cutting and become geometrically visible when the kirigami structure M' is opened. Each hole is an empty region bounded by cut edges, and its geometry fully determines the uniform deployability of M' . Recall that a kirigami structure is *deployable* if each hole can be embedded in the plane while preserving the exact geometry of its neighboring faces. It is *uniformly deployable* if, in addition, all hole opening angles are equal. For example, in Fig. 8 (b), the yellow region is one such hole. In this embedding, the faces adjacent to the hole remain rigid to their counterparts in the

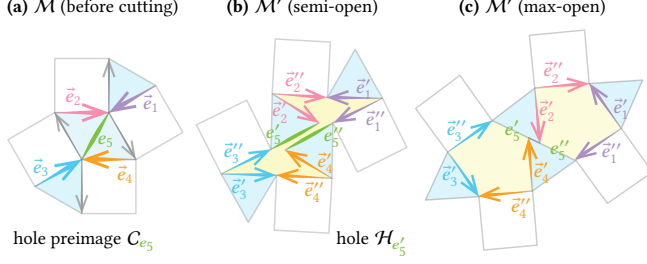


Fig. 8. **Hole detection.** An interior edge is duplicated into two after cutting, connected either by a hinge ($\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$) or completely separated ($\vec{e}_5$). The corresponding edges before cutting (a) and after cutting (b,c) are shown with consistent colors. A single hole that emerges after cutting (highlighted in yellow in b,c) is bounded by a set of hinge vertices and a set of edges that were originally connected before cutting.

original mesh (a), fulfilling the deployability condition. Moreover, the opening angles ($\theta_{\vec{e}_i} = \angle(\vec{e}'_i, \vec{e}''_i)$) are equal, indicating uniform deployability.

Method overview. In Sec. 4.1, we describe how to identify holes for a planar graph with prescribed orientations. Specifically, we detect the closed loops of edges in M' that bound empty regions after deployment, using only the topological structure of M' . Each hole is then mapped back to its corresponding set of interior edges in the original, uncut mesh M . This perspective also provides practical guidance for selecting face-orientation assignments. In Sec. 4.3, we establish geometric conditions on the embedding of the uncut mesh M under which the resulting holes admit planar embeddings with equal opening angles, an equivalent characterization for uniform deployability. In Sec. 4.4, we mathematically analyze the solution space defined by these conditions, characterizing the degrees of freedom of embeddings of a planar graph with a prescribed face orientation that yield uniformly deployable kirigami structures.

4.1 Detecting holes from cutting operations

Applying the cutting operation to all interior edges of M produces a kirigami structure M' . Each interior edge e in M is duplicated after cutting, resulting in two corresponding edges in M' denoted e' and e'' . If $e \in \mathcal{E}_{split}$, then e' and e'' are fully separated; if $e \in \mathcal{E}_{hinge}$, e' and e'' remain connected at a hinge vertex.

Definition 4.1 (Holes in M'). A hole in M' is a connected component of the complement of M' in the plane, whose boundary is a simple closed cycle composed of duplicated interior edges and hinge vertices introduced by the cutting process of M . For any edge $e' \in M'$ that lies on such a boundary, we denote by $\mathcal{H}_{e'}$ the unique hole whose boundary contains e' . See Fig. 8 (b,c).

Definition 4.2 (Hole preimage in M). For an interior edge e in M , let e' in M' be one of its duplicates after cutting. The *hole preimage* C_e is the set of interior edges in M whose corresponding edges in M' collectively form the boundary of the hole $\mathcal{H}_{e'}$.

Although holes exist only in the kirigami structure M' after cutting, each hole admits a unique characterization in terms of its hole preimage in the original, uncut mesh M . Accordingly, instead of

detecting holes directly in M' , we identify hole preimages in M , which allows hole detection to be carried out purely combinatorially on M , without explicit geometric reasoning in M' . To compute the hole preimage C_e associated with an interior edge e in M , we iteratively collect all interior edges of M whose induced cuts contribute to the same hole boundary in M' . Specifically, two adjacent interior edges e_1, e_2 in M belong to the same hole preimage if and only if one of the following situations holds: (1) If both e_1, e_2 are split-cut edges ($e_1, e_2 \in \mathcal{E}_{split}$), they always lie on the same hole boundary. (2) If both e_1, e_2 are hinge-cut edges ($e_1, e_2 \in \mathcal{E}_{hinge}$), they must share a common *target* endpoint (i.e., both edges point toward the shared vertex). (3) If one is a hinge-cut edge and the other a split-cut edge, they belong to the same hole preimage precisely when the hinge-cut edge is directed toward their shared vertex.

These conditions follow directly from the structure of the cutting operation. A hinge-cut edge leaves its two duplicates connected at the target vertex, so the hole boundary passes through that vertex; a split-cut edge separates the two duplicates completely, so both sides of the cut become part of the hole boundary. We can recursively check the neighboring edges and collect all pairs satisfying the above conditions to detect the hole preimages. See Algorithm 1 for the full procedure.

4.2 Face orientation assignment

The hole-preimage analysis above reveals how the face orientation σ governs hole structure: split-cut edges merge otherwise local holes into larger ones, so the size of each hole is proportional to the number of split edges it contains. For auxetic behaviors, we empirically favor many balanced, small holes rather than a few large ones. This motivates choosing face orientations that minimize the number of split edges, i.e., assign opposing orientations to adjacent faces whenever possible. Equivalently, this corresponds to making the dual graph as close to two-colorable as possible.

Consider a planar graph $M = (X, \mathcal{F})$ with a 2D embedding X and a face set $\mathcal{F}$. We would like to assign an orientation to each face:

ALGORITHM 1: Hole Preimage Detection

Input : $\vec{e}_0 \in \mathcal{E}_{hinge}$
Output: the hole preimage C containing $\vec{e}_0$

```

1  $C \leftarrow \{\vec{e}_0\}$ ,  $Q_{toCheck} \leftarrow \{\vec{e}_0\}$ ,  $Q_{checked} \leftarrow \emptyset$ 
2 while  $Q_{toCheck}$  not empty do
3    $e' \leftarrow \text{dequeue}(Q_{toCheck})$ 
4    $\mathcal{N} \leftarrow \text{neighboring\_edges}(e') \setminus (\mathcal{E}_{border} \cup Q_{checked})$ 
5   foreach  $e \in \mathcal{N}$  do
6     if  $e \in \mathcal{E}_{split}$  then
7        $C \leftarrow C \cup \{e\}$ ,  $Q_{toCheck} \leftarrow Q_{toCheck} \cup \{e\}$ 
8     if  $\vec{e} \in \mathcal{E}_{hinge}$  then
9        $\vec{e}$  is directed and denote its direction as  $(v_i \rightarrow v_j)$ 
10      if  $\vec{e} \cap e' = v_j$  then
11         $C \leftarrow C \cup \{\vec{e}\}$ 
12   $Q_{checked} \leftarrow Q_{checked} \cup \mathcal{N}$ 
13 return  $C$ 

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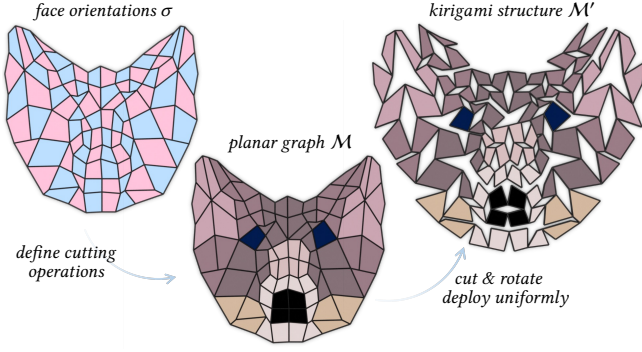


Fig. 9. For the aperiodic pattern shown in the *middle*, we can automatically compute a face-orientation assignment that minimizes the number of split-cut edges (*left*, blue = clockwise, pink = counterclockwise). After optimization, the resulting kirigami structure M' is *uniformly deployable*.

$\sigma : \mathcal{F} \rightarrow \{-1, 1\}$. There are $2^{|\mathcal{F}|}$ possible assignments, but most lead to degenerated kirigami structures. For example, assigning the same orientation to all faces causes complete separation after cutting (as defined in Sec. 3), resulting in a meaningless structure.

We therefore seek orientation assignments that produce a well-formed kirigami pattern M' with auxetic behavior. Our algorithm is guided by the following principles: (1) After cutting, the graph should remain a single connected component. (2) Cut-induced holes should remain local and small. The first condition rules out any face whose neighbors all share the same orientation, as such a face would be fully detached after cutting. The second condition reflects the fact that split-cut edges merge local holes into larger ones. Minimizing and evenly distributing split-cut edges keeps holes small and the structure connected. These goals can be formulated as a max-cut problem on the dual graph. Let $M^d = (\mathcal{V}^d, \mathcal{E}^d)$ denote the dual graph of M , where each vertex in $\mathcal{V}^d$ corresponds to a face in $\mathcal{F}$, and two vertices in $\mathcal{V}^d$ are connected by an edge in $\mathcal{E}^d$ if the corresponding faces in M are adjacent to each other. Minimizing split-cut edges in M is equivalent to making M^d as close to two-colorable as possible. Specifically, we wish to partition $\mathcal{V}^d$ into two sets (corresponding to clockwise and counterclockwise face orientations), to maximize the number of edges between them (i.e., having more pairs of adjacent faces in opposite orientations). Since max-cut is NP-hard, we use the following approximation, based on the algorithm proposed by Goemans and Williamson [1995]. We compute an embedding X^d of M^d , where each vertex lies on the unit circle, by solving the following continuous problem that maximizes the spherical distance between adjacent vertices:

$$\min \sum_{(i,j) \in \mathcal{E}^d} \mathbf{x}_i \cdot \mathbf{x}_j, \quad \text{s.t. } \|\mathbf{x}_i\| = 1, \quad \forall i \in \mathcal{V}^d. \quad (1)$$

A diameter then partitions the circle into two sets, yielding the orientation assignment. For all tested patterns, this algorithm produces well-formed kirigami structures. See Fig. 9 for an example.

4.3 Conditions for uniform deployability

Recall that a kirigami structure $M' = (X', \mathcal{F}')$ is called uniformly deployable if its faces in $\mathcal{F}'$ can rotate *rigidly* about hinge vertices

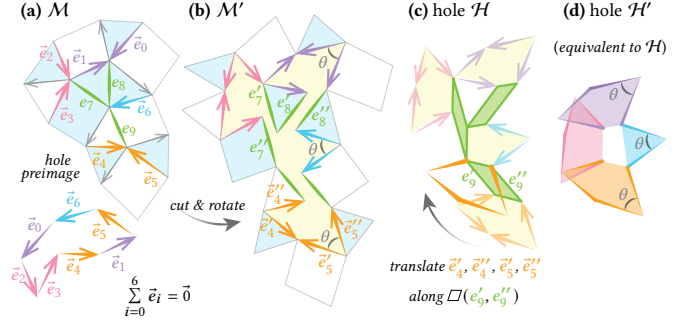


Fig. 10. **Deployability conditions.** The hole preimage C , consists of ten highlighted interior edges in M , $\{\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5, \vec{e}_6, \vec{e}_7, \vec{e}_8, \vec{e}_9\}$ shown in (a), defines a single hole $\mathcal{H}$ (colored in yellow) in the resulting kirigami structure M' shown in (b,c). The uniform deployability of M' is determined by whether the hole $\mathcal{H}$ can be embedded in 2D with equal opening angles. This condition is equivalent to the planarity of the transformed hole $\mathcal{H}'$ in (d): the regions induced by hinge cuts are translated along the parallelograms defined by the neighboring split cuts. See Sec. 4.3 for more details.

without colliding with neighboring faces and all hinge opening angles are equal. Equivalently, uniform deployability requires the existence of a one-parameter family of embeddings $M' = (Y_\theta, \mathcal{F}')$, parameterized by a nonzero opening angle θ , such that each face undergoes a rigid motion and every hinge corresponding to a hinge cut opens by the same angle θ . Fig. 10 (b) shows a uniformly deployable kirigami structure and one of its deployed configurations.

Under this assumption, the deployability of M' is fully determined by the geometry of its holes. In particular, each hole in M' must admit a planar embedding that simultaneously satisfies the angle constraints imposed by the hinges and the edge length constraints imposed by face rigidity. One important design choice in our formulation is to express these geometric conditions in terms of the uncut mesh M , which guarantees that the resulting kirigami structure M' is fully closed and gap-free before deployment.

Proposition 4.1. A kirigami structure M' is uniformly deployable if and only if for every interior edge e_0 in the original tiling M , its associated hole preimage C_{e_0} satisfies

$$\sum_{\vec{e} \in C_{e_0} \cap \mathcal{E}_{\text{hinge}}} \vec{e} = \vec{0}, \quad (2)$$

that is, the directed edges corresponding to hinge cuts in the hole preimage sum to the zero vector.

PROOF. Under the uniform deployability assumption, the induced edges in M' that correspond to a *split cut* in M remain *parallel* during deployment at any opening angle; see Appendix A for a detailed proof. As a consequence, the regions adjacent to split cuts form parallelograms and are always planar (Fig. 10 b,c). To determine whether a single hole $\mathcal{H}$ can be properly embedded, it suffices to examine the regions induced by hinge cuts. Sliding these regions along the parallelograms of the split cuts produces a transformed polygon $\mathcal{H}'$ (Fig. 10 d). The original hole $\mathcal{H}$ in M' admits a valid planar embedding if and only if this transformed polygon $\mathcal{H}'$ is planar. Checking the planarity of $\mathcal{H}'$ is a solved problem. The polygon

$\mathcal{H}'$ is a $2k$ -gon with edge lengths $\{\ell_0, \ell_0, \dots, \ell_{k-1}, \ell_{k-1}\}$ from the hinge cuts and internal angles at the hinge vertices equal to θ . Here k is the number of directed edges (hinge cuts) corresponding to the hole preimage in $\mathcal{M}$. Its planarity is equivalent to the vector-sum condition in Eq. (2) [Segall et al. 2025, Appendix B]. $\square$

Fig. 10 shows an example where the uniformly deployable condition, Eq. (2), is satisfied. In contrast, Fig. 11 shows a periodic pattern (the bold lines indicate periodicity) in which one hole preimage $\mathcal{C}_1$ satisfies the condition, while the other $\mathcal{C}_2$ does not. Consequently, the embedding in Fig. 11, with the indicated face-orientation assignment, does not admit a uniformly deployable kirigami structure. Since Eq. (2) is linear in the edge vectors, it immediately follows that:

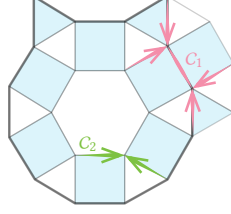


Fig. 11. Non-deployable

Remark 4.1. Uniform deployability is *invariant* under affine transformations: any affine transformation of a uniformly deployable kirigami structure remains uniformly deployable.

4.4 Shape space analysis: Tutte auxetic embeddings

Given a planar graph with prescribed face orientations, each hole in the resulting kirigami structure $\mathcal{M}'$ imposes a linear geometric constraint on the embedding of the original mesh $\mathcal{M}$, as described by Eq. (2). In other words, analyzing uniform deployability is equivalent to finding vertex positions $\mathbf{X}$ for the uncut mesh $\mathcal{M}$ such that all constraints on hole preimages are satisfied simultaneously:

$$\sum_{(v_i \rightarrow v_j) \in C} (\mathbf{x}_j - \mathbf{x}_i) = \mathbf{0}, \quad \forall C \in \mathcal{C}^{\mathcal{M}}, \quad (3a)$$

$$\mathbf{x}_p - \mathbf{x}_{p'} = \mathbf{t}_h, \quad \forall (p, p') \in \mathcal{I}_h, \quad (3b)$$

$$\mathbf{x}_q - \mathbf{x}_{q'} = \mathbf{t}_v, \quad \forall (q, q') \in \mathcal{I}_v. \quad (3c)$$

Here, $\mathcal{C}^{\mathcal{M}}$ denotes the complete set of hole preimages of $\mathcal{M}$; for each C we consider only hinge-cut edges, represented as directed edges $(v_i \rightarrow v_j)$; $\mathbf{x}_i \in \mathbb{R}^2$ is the unknown position of the i -th vertex in $\mathcal{M}$. The term in Eq. (3a) enforces the geometric condition for uniform deployability defined in Eq. (2). Boundary vertices may either be fixed to predefined positions or constrained to be *periodic* (as specified in Equations (3b) and (3c)), allowing the pattern to tile the plane and generate structures of arbitrary size. The index sets $\mathcal{I}_h$ and $\mathcal{I}_v$ contain pairs of vertices that are periodic in the horizontal and vertical directions (either specified manually or detected automatically from a periodic embedding of the input planar graph); $\mathbf{t}_h$ and $\mathbf{t}_v$ define the corresponding periodicity offsets. Note that $\mathbf{t}_h$ and $\mathbf{t}_v$ can be any linearly independent vectors, and the resulting embeddings are equivalent up to affine transformations, as per Remark 4.1.

Revisiting the Tutte embedding. This formulation naturally connects to the classical Tutte embedding [Tutte 1963], well known for embedding planar graphs. Tutte's method solves the embedding $\mathbf{y}_i$ for each interior vertex i that satisfies: $\sum_{j \in \mathcal{N}(i)} (\mathbf{y}_i - \mathbf{y}_j) = \mathbf{0}$, where $\mathcal{N}(i)$ denotes the set of vertices adjacent to i . Boundary vertices are often prescribed on a convex shape (e.g., a circle) to avoid

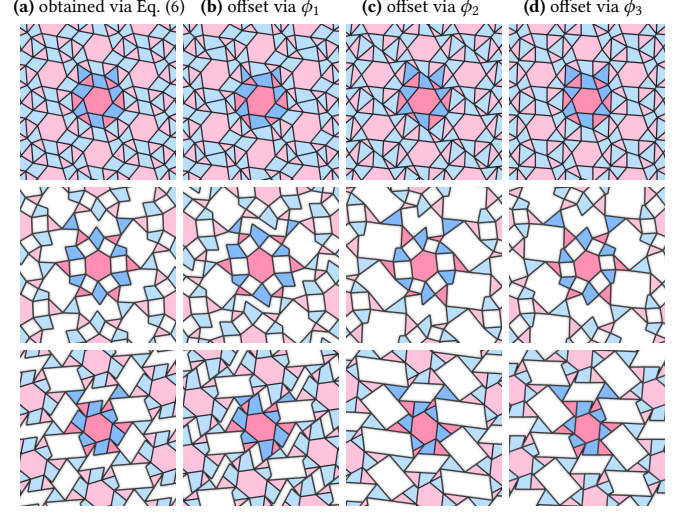


Fig. 12. **Top:** four Tutte auxetic embeddings for the graph shown in Fig. 11; their periodic unit patterns are highlighted in darker shades. **Middle:** the resulting kirigami structures deployed with identical opening angles. **Bottom:** deployment at the maximum opening configuration.

degenerated embeddings. This problem can be reformulated as a linear system $\mathbf{A}\mathbf{Y} = \mathbf{b}$, combining constraints for interior vertices (encoded by the graph Laplacian) with boundary constraints. For a connected graph, this system is full rank and has a unique solution.

Tutte auxetic embedding. In our setting, the problem in Eq. (3) is a linear system, in close analogy to the classical Tutte embedding:

$$\begin{pmatrix} \mathbf{L}_{H \times N} \\ \mathbf{B}_{B \times N} \end{pmatrix} \mathbf{X}_{N \times 2} = \begin{pmatrix} \mathbf{0}_{H \times 2} \\ \mathbf{T}_{B \times 2} \end{pmatrix}, \quad (4)$$

where N is the total number of vertices, B is the number of boundary vertices, H is the number of hole preimages in $\mathcal{M}$. The matrix $\mathbf{L}$ is constructed analogously to the graph Laplacian, but its rows encode vector-sum constraints over hinge-cut edges belonging to the same hole preimage, as specified in Eq. (2). The matrices $\mathbf{B}$, $\mathbf{T}$ encode boundary conditions corresponding either to fixed boundary vertices or to periodic constraints defined in Equations (3b) and (3c). In the periodic boundary setting, boundary vertices may appear unconstrained; however, all periodic configurations are equivalent up to global translations or affine transformations. Since such transformations do not affect uniform deployability (Remark 4.1), boundary vertices can be fixed to any periodic configuration. A solution of Eq. (4) is called a *Tutte auxetic embedding*. Geometrically, the system captures the relation between interior vertices and holes. The number of unknowns equals the number of interior vertices, while the number of independent equations (excluding the boundary constraints) equals the number of holes in $\mathcal{M}'$, which is *at most* the number of interior vertices, since adjacent split cuts merge multiple edges into a single hole. When no split-cut edges exist ($\mathcal{E}_{split} = \emptyset$), each interior vertex corresponds to a distinct hole, and the system is full rank with a unique solution; this occurs precisely when the graph is two-colorable and any two adjacent faces have opposite

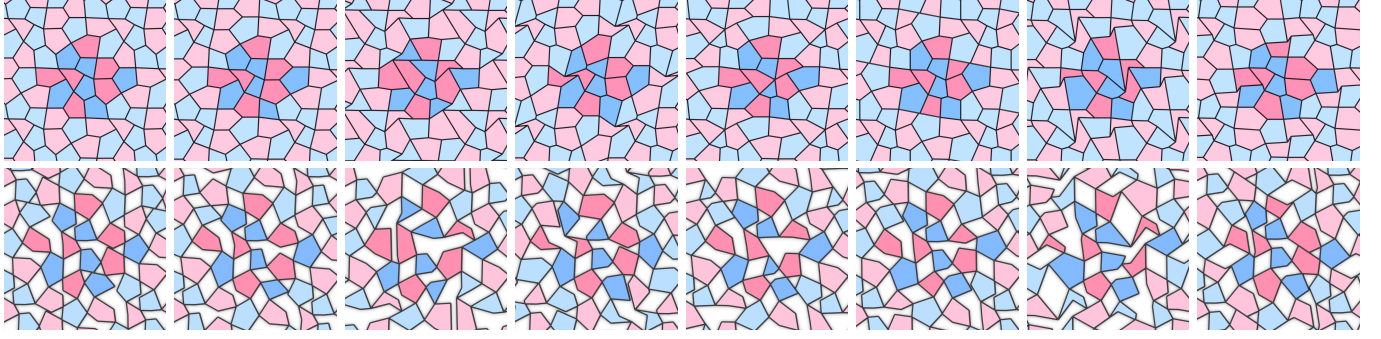


Fig. 13. This planar graph, with the illustrated face orientations, has a full space of Tutte auxetic embeddings with eight degrees of freedom. Each column shows the perturbation along a single basis vector of this space. The *top* row shows the embedding before cutting, and the *bottom* row shows the deployment at an opening angle of 60° .

orientations. Otherwise, the system becomes rank-deficient, and its null space describes the shape space of Tutte auxetic embeddings.

Shape space analysis. The linear system Eq. (4) provides an analytic framework to explore the complete set of Tutte auxetic embeddings, denoted as $\mathcal{X}$. When the system is full rank, the solution is unique. Rank deficiencies, however, reveal a multi-dimensional shape space of solutions, allowing exploration of geometrically distinct yet uniformly deployable kirigami patterns. We can perform a singular value decomposition (SVD) of the system matrix in Eq. (4) to identify the null space, collecting the vectors $\{\phi_i\}_i$ corresponding to zero singular values. Each ϕ_i identifies a set of vertices that can be translated *together* without violating the deployability constraints. Using this, the entire space of Tutte auxetic embeddings can be expressed explicitly as:

$$\mathcal{X} = \left\{ \mathbf{X}_0 + \sum_i \phi_i \mathbf{t}_{\phi_i}^T, \forall \mathbf{t}_{\phi_i} \in \mathbb{R}^2 \right\}, \quad (5)$$

where $\mathbf{t}_{\phi_i}$ are the coefficients (for the x and y coordinates) of the eigenvector ϕ_i of the kernel and $\mathbf{X}_0$ is any particular Tutte auxetic embedding. One way to obtain such an $\mathbf{X}_0$ is via projection:

$$\min_{\mathbf{X}} \|\mathbf{X} - \mathbf{X}^{\text{ini}}\|^2 \quad \text{s.t. } \mathbf{X} \text{ satisfies Eq. (4)}. \quad (6)$$

That is, $\mathbf{X}_0$ is the closest embedding to the initial guess that fulfills the deployability constraints (see Figures 12 and 14). Vertices corresponding to each null-space basis vector ϕ_i can be translated collectively in any direction, producing new embeddings that preserve uniform deployability. This formulation gives a complete, constructive characterization of the shape space, enabling systematic exploration of diverse auxetic configurations. For example, the Tutte auxetic embeddings of the planar graph in Fig. 11 span a six-dimensional shape space, with basis vectors $\{\phi_i\}_{i=1,2,3}$. Perturbing its auxetic embedding shown in Fig. 12 (a) along each basis generates alternative embeddings (Fig. 12 b,c,d) that are periodic and uniformly deployable, yet geometrically distinct, as evidenced by the different hole behaviors during deployment. Fig. 13 shows a more complicated pattern with an eight-dimensional shape space.

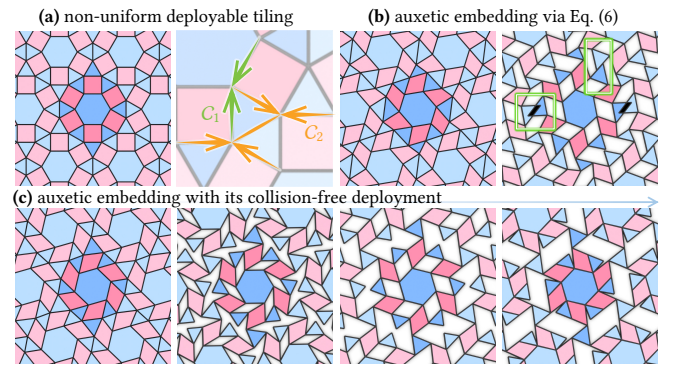


Fig. 14. (a) A hole preimage C_1 that violates Eq. (2). (b) The modified tiling is auxetic but has a limited deployment range due to collisions (highlighted in green boxes) at small opening angles. (c) A new auxetic embeddings that is collision-free ($\theta_{\max} = \pi$).

4.5 Collision handling during shape optimization

In theory, one may translate along any eigenvector ϕ_i with arbitrary offsets $\mathbf{t}_{\phi_i}$, and the resulting embedding remains uniformly deployable. In practice, however, such embeddings may exhibit undesirable geometric properties, including self-intersections or a severely limited deployment range. At the extreme, choosing $\mathbf{t}_{\phi_i} \rightarrow \infty$ yields an embedding that is mathematically valid but practically useless. Small offsets can still produce geometries with restricted deployment, where collisions occur at small opening angles (e.g. Fig. 14 (b), Fig. 15 (e)), thereby limiting the applicability of the resulting patterns. We provide a solution for collision handling within the Tutte auxetic embedding shape space $\mathcal{X}$.

The key observation is that collisions, when they occur, arise immediately after some pair(s) of edges becomes collinear, as illustrated in Fig. 15 (b); beyond this point, further rotation forces the edges into collision. We begin by mathematically characterizing when this situation occurs. Consider a uniformly deployable kirigami structure $\mathcal{M}' = (\mathbf{Y}^\theta, \mathcal{F}')$, where vertex positions along the deployment process are parameterized by the opening angle θ . For any given θ , the configuration $\mathbf{Y}^\theta$ can be computed explicitly via forward kinematics, since each face undergoes a rigid rotation by θ

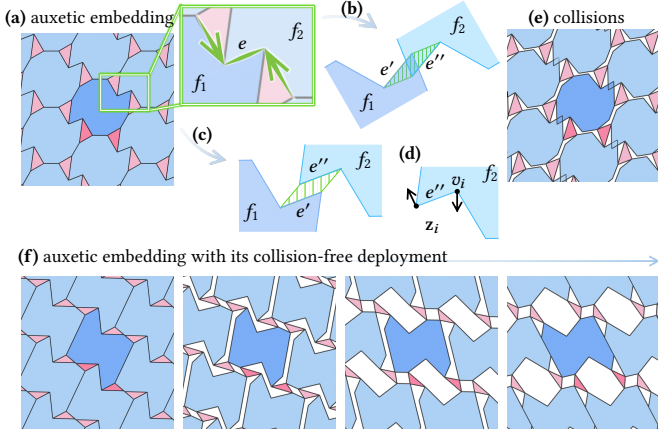


Fig. 15. The auxetic embedding shown in (a) yields a kirigami structure with limited deployment range, where collisions occur at a small opening angle, as shown in (e). Collisions arise only when the duplicates of a split-cut edge become colinear during deployment. We highlight the split-cut edge e between faces f_1 and f_2 , along with its duplicates e' and e'' created by the cutting process. Two deployed configurations are shown in (b) and (c), illustrating cases with and without collisions, respectively. To prevent collisions, the deployed edges e'' and e' are encouraged to be displaced away from faces f_1 and f_2 , respectively. This is formulated as an energy defined on gradients with respect to vertex positions, illustrated in (d). The resulting optimized auxetic embedding, exhibiting collision-free deployment, is shown in (f).

relative to the initial embedding Y_0 . Let $g(Y^\theta, \alpha)$ denote deploying the kirigami structure from the configuration Y^θ by an additional angle α , where g is built analytically using forward kinematics. This is equivalent to $g(Y^\theta, \theta + \alpha)$, i.e., deploying the initial embedding by the total opening angle $\theta + \alpha$. This notation enables us to examine the deployed configuration at any opening angle θ and determine whether collisions will occur under further deployment. This can be captured by

$$\left. \frac{\partial g(Y^\theta, \alpha)}{\partial \alpha} \right|_{\alpha=0}, \quad (7)$$

which corresponds to infinitesimally opening the current configuration Y^θ . The i -th row of the derivative in Eq. (7), denoted by z_i for notational simplicity, captures the relative position change of the i -th vertex of M' under an infinitesimal opening angle. We focus on the edge $e' = (v_i, v_j) \in M'$ that is colinear to another e'' at the current deployment angle θ . Its endpoints v_i and v_j have positions y_i^θ, y_j^θ at current deployment configuration. The vector z_i gives the 2D offset corresponding to the infinitesimal displacement of vertex v_i when M' is further opened. Collisions can be detected by examining the direction of z_i . If z_i points into face f_1 (the face to which e' belongs), then further opening will not cause a collision with the neighboring face f_2 , see Fig. 15 (c). In contrast, if z_i points toward the neighboring face f_2 , the edge is translated into f_2 , leading to collisions upon further deployment, see Fig. 15 (b). This can be quantified by

$$\mathcal{B} \left(z_i \cdot \left(\frac{y_i^\theta - y_j^\theta}{\|y_i^\theta - y_j^\theta\|} \right)^\perp \right), \quad (8)$$

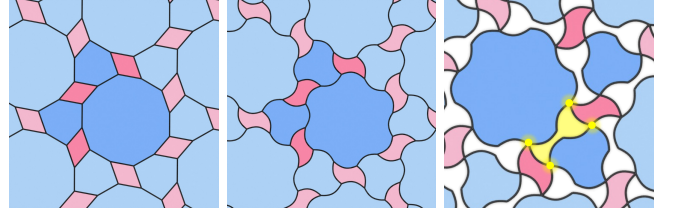


Fig. 16. Left/Middle: An auxetic embedding with straight/curved cuts. Right: the pattern is deployable if the holes can be embedded in the plane, which depends only on the positions of the corner points, not on the paths connecting them. That is, the yellow region is planar as long as the four highlighted points are coplanar.

where $\mathcal{B}$ is a soft barrier function that assigns a high penalty to negative values (i.e., when z_i points toward the neighboring face f_2) and vanishing penalty to positive values (i.e., when z_i points toward its own face f_1).

Since the deployment range depends on the initial embedding Y^0 , and we observe in all experimental cases that collisions occur along the split-cut edges, we choose to solve the following optimization problem:

$$\min_{Y \in \mathcal{X}} \sum_{(v_i, v_j) \in \mathcal{E}_{split}} \mathcal{B} \left(z_i(Y) \cdot \left(\frac{y_i - y_j}{\|y_i - y_j\|} \right)^\perp \right) + \gamma \|Y - Y^0\|. \quad (9)$$

Here, we directly optimize the embedding Y before deployment (i.e., choose $\theta = 0$) to minimize the potential for collisions along the split-cut edges, with the additional constraint that Y remains auxetic, together with the regularizer to encourage proximity to the initial embedding. Note that $\mathcal{X}$ and $\mathcal{E}_{split}$ are formally defined only in the uncut mesh M , but we abuse notation and transfer it to M' for convenience: by $(v_i, v_j) \in \mathcal{E}_{split}$ we mean that the edge (v_i, v_j) in M' corresponds to a split-cut edge in M , and by $Y \in \mathcal{X}$ we mean the embedding Y of M' before deployment (the configuration contains no geometric holes, only topological holes) satisfies the uniform deployability of the uncut mesh.

4.6 Tutte auxetic embeddings with curved cuts

Uniform deployability depends only on the positions of the hinge vertices that define the hole, not on the path connecting them (see Fig. 16). Consequently, it is possible to replace the straight edges of a uniformly deployable embedding with curved edges without affecting the uniform deployment property. Specifically, for any hinge edge $e \in \mathcal{E}_{hinge}$, if we fix the frame of reference to one of its adjacent faces, each vertex on the duplicated edge moves along a circle centered at the hinge vertex, with radius equal to its distance from the hinge. As long as the circular trajectory of each point intersects the original edge curve at exactly one point, the edge can rotate around the hinge vertex without causing collisions. Therefore, a straight hinge edge can be replaced by a curve $C(t)$ that starts at the hinge vertex v_h , ends at the other edge vertex v_e and has a distance to the hinge vertex that is monotonically non-decreasing:

$$C(0) = v_h, \quad C(1) = v_e, \quad \frac{d}{dt} \|C(t) - v_h\| \geq 0. \quad (10)$$

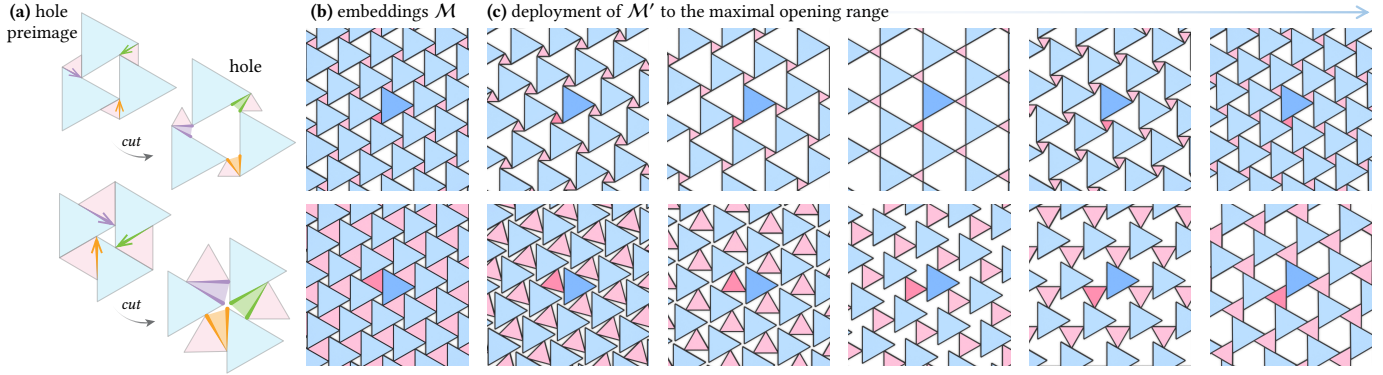


Fig. 17. **Periodic embeddings with interior boundaries.** Holes initially present in $\mathcal{M}$, see (b), are planar by construction; after cutting, they expand by merging with holes created by partial or split cuts, see (c). The planarity analysis of $\mathcal{M}'$ remains unchanged (Sec. 4.3): it depends only on hole regions associated with partial-cut edges, leading to the same geometric condition illustrated in (a).

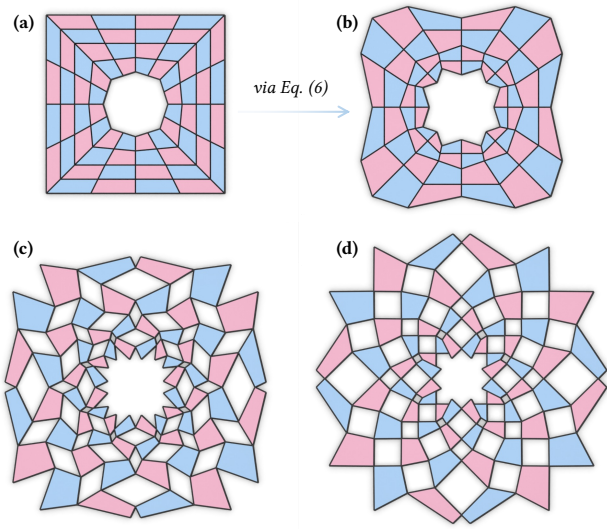


Fig. 18. **Aperiodic embedding with interior boundaries.** (a) Initial embedding. (b) Optimized embedding. (c) Intermediate deployed state. (d) Fully deployed state.

Note that, unlike straight-line cuts, curved cuts are more prone to collision and may require dedicated optimization to ensure collision-free with large range of deployment [Liu et al. 2024].

5 Results

Our analysis applies to *arbitrary* planar graphs, including those that are non-periodic (Figures 1 and 3), non-2-colorable (Fig. 1, 9 and 11) or with inner holes (Figures 17 and 18). Our formulation is flexible and enables the design of multiple distinct cutting patterns for the same planar graph within the same geometry (Figures 6 and 7). Furthermore, uniform deployability depends only on the positions of the endpoints of the partial-cut edges and not on the path connecting them, allowing for the generation of patterns with curved cuts (Figures 19 and 20). Additionally, we characterize the full space of auxetic embeddings (Figures 12 and 13), enabling easy

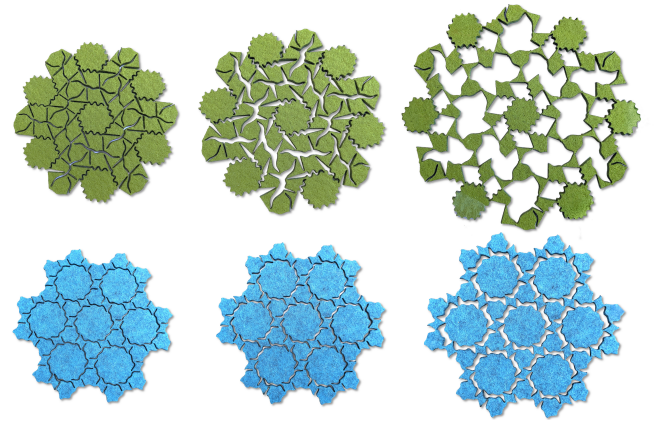


Fig. 19. Deployment of two patterns with curved cuts, fabricated using felt.

modification of a non-auxetic embedding with minimal changes (Figures 12 and 21). In contrast, the baseline method of Segall et al. [2025] is limited to 2-colorable planar graphs and can only verify whether a given embedding of such a graph is auxetic. Furthermore, for a valid two-colorable embedding, their cutting operation is fixed, producing only a single valid kirigami pattern.

The explicit characterization of the complete set of Tutte auxetic embeddings $\mathcal{X}$ enables the design of kirigami patterns with target properties, such as predefined negative Poisson ratios at specific opening angles (Sec. 5.1) or patterns that exhibit two fully closed configurations during deployment (Sec. 5.2). Sec. 5.3 provides the details for inverse design using our newly discovered patterns. Sec. 5.4 provides the implementation details for our web UI for pattern exploration and optimization.

5.1 Isotropic kirigami patterns with conformal deployment

One particularly challenging scenario is the design of *isotropic* auxetic materials with a Poisson ratio of -1 at any opening angle that is invariant under global rotation of the pattern. In other words, the deployment process exhibits conformal behavior. Previous works either rely on well-known isotropic structures [Konaković et al.

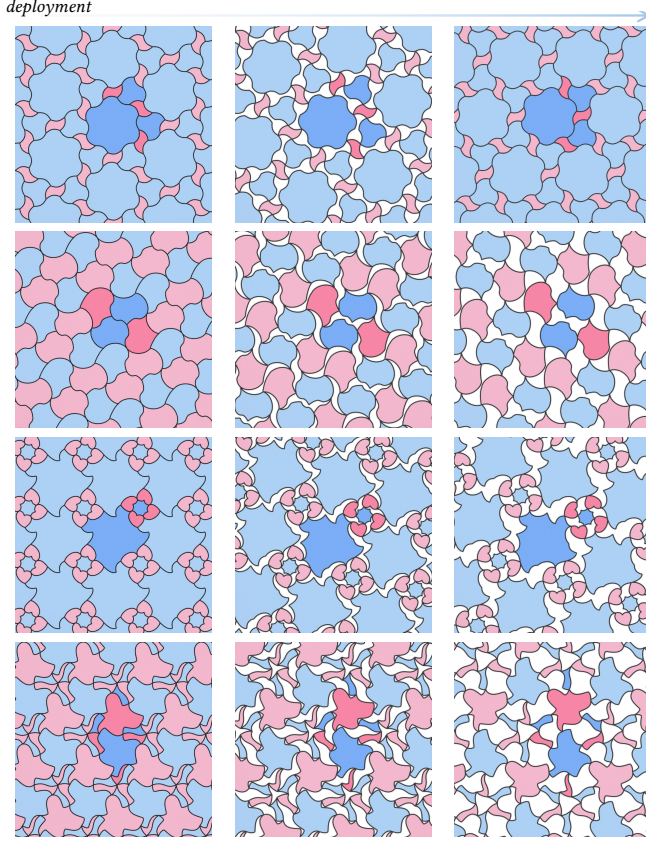


Fig. 20. Four examples of uniformly deployable kirigami structures featuring distinct curved cuts. See Appendix 4.6 for more details.

2016, 2018] or experimentally explore new embeddings of fixed structures [Shan et al. 2015]. Our framework allows us to solve for isotropic auxetic embeddings of a given kirigami pattern, as shown in Fig. 21. We focus on *periodic* kirigami structures $\mathcal{M}'(\mathbf{Y}^\theta, \mathcal{F}')$, obtained from solving the geometric conditions for uniform deployability with periodic boundary conditions.

Our approach builds on the following observations: **(1)** If a uniformly deployable kirigami is derived from a periodic tiling, then any deployment configuration $\mathbf{Y}^\theta$ remains periodic for all opening angles θ . That is, $\mathbf{Y}^\theta$ together with the holes forms a 2D periodic tiling. **(2)** Any periodic tiling can be represented as a tiling with the wallpaper group $p1$ by grouping a finite number of original tiles into a metatile, see Fig. 22. This presentation preserves the deployment behavior while reducing the symmetry description to pure translation. **(3)** The two translation symmetry vectors of the metatile form a parallelogram. The metatile and its translation symmetry vectors change during the deployment process. **(4)** The global deployment behavior of the kirigami $\mathcal{M}'$ can therefore be characterized entirely by the evolution of this parallelogram. In particular, conformal deployment corresponds to transformations in which the parallelogram undergoes only uniform scaling and rotation, with its shape preserved.

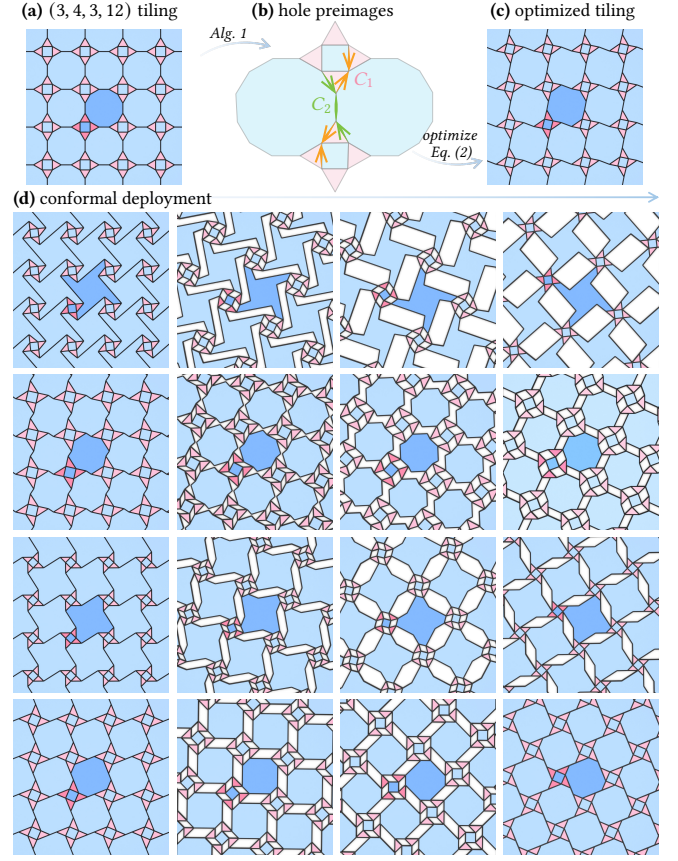


Fig. 21. The Euclidean tiling (3,4,3,12) shown in (a) is not uniformly deployable, as the hole preimage C_1 violates Eq. (2), highlighted in (b). The optimized tiling in (c) is uniformly deployable, but its deployment exhibits varying Poisson ratios. We then explore the design space of uniformly deployable embeddings with the same topology and identify conformal deployments; four representative examples are shown in (d), one per row.

Let $\mathbf{p}_x^\theta$ and $\mathbf{p}_y^\theta$ denote the translation vectors of the periodic tiling induced by the embedding $\mathbf{Y}^\theta$ at opening angle θ (see Fig. 21). We assemble these vectors into matrices:

$$\mathbf{P}_0 = \begin{pmatrix} \mathbf{p}_x^0 & \mathbf{p}_y^0 \end{pmatrix}, \quad \mathbf{P}_\theta = \begin{pmatrix} \mathbf{p}_x^\theta & \mathbf{p}_y^\theta \end{pmatrix}. \quad (11)$$

The deformation of the fundamental parallelogram is captured by the Jacobian:

$$\mathcal{J}(\theta) = \mathbf{P}_\theta \mathbf{P}_0^{-1} = \begin{pmatrix} J_{11}(\theta) & J_{12}(\theta) \\ J_{21}(\theta) & J_{22}(\theta) \end{pmatrix}. \quad (12)$$

Conformal deployment requires $\mathcal{J}(\theta)$ to be a similarity matrix for all θ . Infinitesimally, this is equivalent to the vanishing derivative of the conformal distortion. We therefore search for an embedding $\mathbf{Y}^0$ using the following optimization:

$$\begin{aligned} \mathbf{Y}^0 &= \arg \min_{\mathbf{Y} \in \mathcal{X}} \|\mathbf{Y} - \mathbf{X}_0\|^2 \\ \text{s.t. } \left. \frac{\partial (J_{11} - J_{22})}{\partial \theta} \right|_{\theta=0} &= 0, \quad \left. \frac{\partial (J_{12} + J_{21})}{\partial \theta} \right|_{\theta=0} = 0, \end{aligned} \quad (13)$$

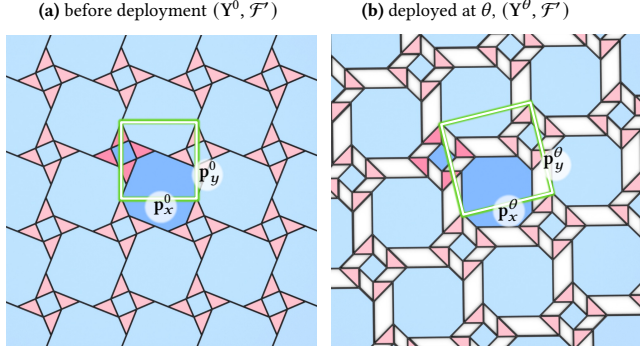


Fig. 22. The fundamental parallelogram (highlighted in green) of the kirigami structure before (a) and after (b) deployment.

where X_0 is the initial embedding of the given pattern. Both the Jacobian and its partial derivatives are linear expressions in the vertex positions Y , hence the optimization in Eq. (13) amounts to solving a linear system. Empirically, we observe that for all tilings with a non-trivial kernel ($\mathcal{E}_{split} \neq \emptyset$), solving Eq. (13) yields an embedding with conformal deployment. Moreover, once the derivative of the conformal distortion vanishes at $\theta = 0$, the deployment remains conformal for all opening angles θ .

5.2 Fully-closed kirigami patterns after deployment

Another interesting kirigami pattern design problem is finding patterns that exhibit two fully closed configurations during deployment, a feature useful for 2D contour approximation [Dudte et al. 2016] or 3D shape approximation [Segall et al. 2025], where both the initial and final configurations are completely closed with no gaps. Previous works rely on preexisting patterns known to possess this property. In contrast, our framework can systematically discover new patterns with this behavior (see Fig. 2). Starting from a gap-free pattern, we solve for an auxetic embedding within X that has a gap-free configuration at the maximum deployment angle $\theta^{\max}$.

We first give a formal definition of the maximum deployment angle. Consider a hinge-cut edge $(v_i \rightarrow v_j) \in \mathcal{M}$. After cutting, the edge is duplicated in the resulting kirigami structure $\mathcal{M}'$, denoted as $(v_i, v_j) \in f_1, (v_i, v_k) \in f_2$, forming a hinge at the source vertex v_i . At deployment angle θ , the kirigami structure $\mathcal{M}'$ has embedding Y^θ , and the angle formed at the hinge y_i satisfies $\angle y_j y_i y_k = \theta$. The maximum opening angle at hinge y_i depends on the geometry of the adjacent faces f_1 and f_2 . We define this maximum opening angle, denoted by β_i , as the largest value for which the two faces do not collide; beyond β_i , further rotation causes collision on their opposite sides. The maximum deployment angle of the kirigami structure $\mathcal{M}'$ is then defined as $\min_i \beta_i$, the smallest admissible opening angle among all hinge vertices. Beyond this angle, at least one pair of faces will collide.

From this definition, we immediately observe that a necessary condition for a fully closed configuration at the maximum deployment angle is that all hinges share the same maximal opening angle, i.e., $\beta_i = \beta, \forall i$. Otherwise, when some face pairs are just about to collide, other face pairs will not, leaving gaps in the maximally deployed configuration. However, this condition is not sufficient: it

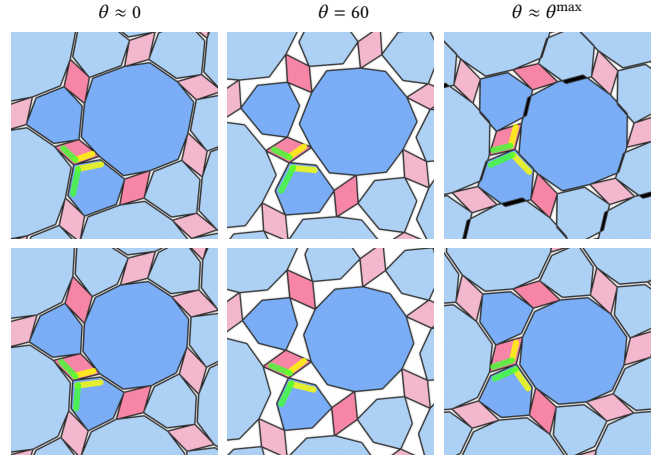


Fig. 23. Fully-closed kirigami pattern $\mathcal{M}'$ before (top) and after (bottom) optimization, with its deployment at three opening angles θ . Yellow highlights a pair of edges connected by a hinge that are co-located at $\theta = 0$ and separate upon deployment. These edges have equal length as they originate from the same split-cut edge of $\mathcal{M}$. Green highlights another pair of edges exhibiting the opposite behavior: they are initially separated at $\theta = 0$ and approach each other during deployment, becoming closer at $\theta^{\max}$. For an arbitrary auxetic embedding (top), the green edges need not have equal length; however if they do (bottom), the configuration at $\theta^{\max}$ can be fully closed, as the edges align perfectly with no gap between them.

may also result in a configuration with no gaps but with overlapping faces. To prevent this, we further require that, when two faces meet on the opposite side, the corresponding edges that come into contact have equal length.

Our optimization energy for achieving a fully closed configuration at the maximal deployment angle thus includes two terms: (1) If two faces share a hinge edge (and the corresponding hinge vertex), the other two edges of the adjacent faces that meet at the hinge vertex should have equal length at the maximum deployment angle. See Fig. 23. (2) The maximum deployment angle should be uniform across all hinges. Let e_{ij} be a hinge edge with hinge vertex v_i , and let e_{li}, e_{ik} denote the two edges in the adjacent faces to e_{ij} that meet at v_i . Let α_{lik} be the angle between these two edges. The maximum deployment angle at the corresponding hinge is $2\pi - \alpha_{lik}$, at which these two edges meet. We minimize the following energy to optimize for a fully closed pattern:

$$\min_{Y \in X} \sum_{(v_i \rightarrow v_j) \in \mathcal{E}_{hinge}} (\|y_l - y_i\| - \|y_i - y_k\|)^2 + (\alpha_{lik} - \bar{\alpha})^2, \quad (14)$$

where $\bar{\alpha}$ is the average of all α_{lik} and is updated after each optimization step. Fig. 2, Fig. 20 (first row) and Fig. 21 (last row) show some of our newly discovered patterns that are fully closed at maximum deployment angle.

5.3 Inverse design

Our framework expands the set of kirigami patterns available for inverse design, enabling target geometry approximation either with a remeshed topology (Fig. 24) or the exact original topology (Figures 3 and 25). We follow Segall et al. [2025] to optimize a chosen deployable kirigami structure such that the distance between its

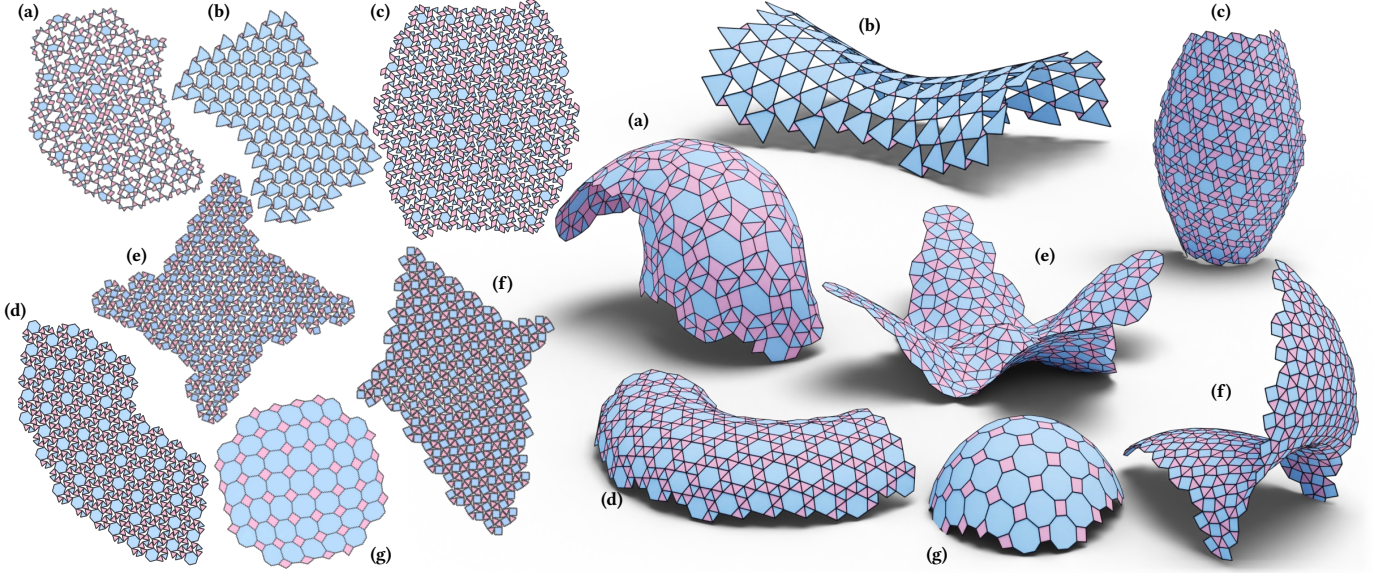


Fig. 24. **Inverse design.** Deploying the semi-open, flat materials (*left*) to their fully closed configurations (*right*) to achieve the target 3D shapes in different structures. Corresponding patterns are labeled in the same letter. Note that for the hemisphere example, the initial configuration is also gap-free, meaning the curvatures are realized solely through combinatorial changes.

deployed stage and the input mesh is minimized. The resulting deployed kirigami structure can be viewed as a remeshed version of the input. See the right side of Fig. 24.

Reproducing the topology (i.e., face connectivity or vertex combinatorics) of the original mesh is more challenging. Given a target shape $S = (X, \mathcal{F})$ with vertices X and polygonal faces $\mathcal{F}$, we treat it as a planar graph $\mathcal{G}$ embedded in 3D. Following Sec. 4.2, we assign face orientations to $\mathcal{F}$ and solve for its Tutte auxetic embedding, i.e. 2D positions X_0 satisfying the uniform deployability conditions in Eq. (2), with the boundary vertices fixed to a circle. Alternatively, one can also first compute a parameterization for the target shape and solve for its Tutte auxetic embedding via Eq. (6). This yields a uniformly deployable kirigami structure $\mathcal{M} = (Y_0, \mathcal{F})$ (e.g., bottom left in Figures 3 and 25), which shares the *topology* of the input. However, $\mathcal{M}$ deploys in-plane. To approximate the input geometry, we seek a semi-open kirigami structure whose fully closed deployment matches the input mesh. We cut $\mathcal{M}$ based on face orientations to obtain $\mathcal{M}' = (Y_\theta, \mathcal{F}')$ at opening angle θ , where Y_θ is known due to uniform deployability. We then adjust Y_θ so that edge lengths in $\mathcal{M}'$ correspond to those in S . The open regions provide freedom for optimization. The resulting optimized structure $\mathcal{M}'$ is shown in the bottom right of Figures 3 and 25. Once they are deployed to a fully-closed configuration, they will reproduce the exact topology of the target shape with the geometry well preserved.

5.4 User interface and implementation

We implement an integrated web-based user interface for designing and exploring Tutte auxetic embeddings, including inverse design. Fig. 26 shows a screenshot of the UI, and interactive demonstrations are provided in the supplementary video. Our UI supports the following main functionalities:

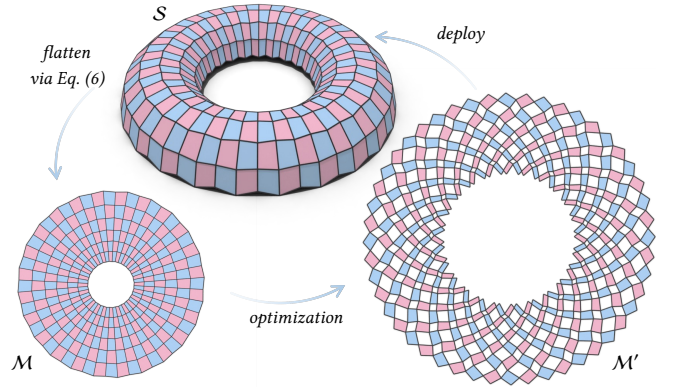


Fig. 25. **Inverse design** of a half torus shape S , where the optimized kirigami structure $\mathcal{M}'$ can reproduce the topology of the original mesh after deployment.

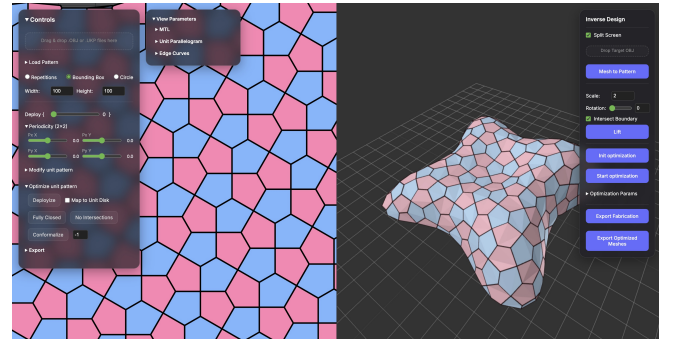


Fig. 26. Our web-based UI that allows for pattern design, shape space exploration, pattern optimization and inverse design.

- Load a planar graph and automatically assign face orientations as described in Sec. 4.2.
- Interactively flip face orientations by clicking on faces.
- Compute the full design space and update a non-uniformly deployable pattern to its closest Tutte auxetic embedding via Eq. (6).
- Adjust the coefficients of the basis vectors using sliders; the pattern updates in real time while remaining uniformly deployable.
- Simulate the deployment process with automatic detection of the deployment range.
- Solve for Tutte auxetic embeddings that are isotropic (Sec. 5.1) or fully-closed after deployment (Sec. 5.2), with automatic collision handling (Sec. 4.5).
- Draw custom curves and assign them to selected cut edges; the pattern updates with curved cuts.
- Load a disk-topology mesh and perform inverse design using the selected pattern.

The core algorithms are implemented in a WebAssembly backend, compiled from C++ code. We optimized all the non-linear energies (discussed in Sec. 4.5, 5.1 and 5.2) via Newton's method with a per-element Hessian projection onto the PSD cone. For periodic patterns, the computational complexity of face assignment depends only on the graph size of its meta-tile. Typically, it takes a few milliseconds for almost all kirigami patterns considered in the paper (where the meta-tile has 3–20 faces) and up to a few hundred milliseconds for graphs with hundreds of faces (e.g., Figures 3 and 25). The performance of the non-linear optimizations within the Tutte shape space $\mathcal{X}$ depends greatly on the dimension of $\mathcal{X}$: it ranges from a few milliseconds when the dimension is below 10 to a few seconds when the dimension is around 100. Overall, our user interface is interactive and efficient, enabling real-time feedback and intuitive exploration. Full implementation can be found at <https://github.com/segaviv/tuttekiri>.

6 Conclusion, limitations & future work

This work provides a general geometric framework for analyzing and constructing uniformly deployable rigid kirigami structures from general planar graphs. By expressing deployability through linear constraints induced by holes created during cutting, we characterize the full space of embeddings that admit uniform deployment. Several limitations suggest promising directions for future work. First, although all embeddings in the solution space $\mathcal{X}$ are theoretically uniformly deployable, some may be geometrically undesirable (e.g., exhibiting extremely short edges or sharp angles). Moreover, uniform deployability does not guarantee a large collision-free deployment range; self-intersections may occur at small opening angles (e.g., Fig. 14). At present, such issues can only be detected through explicit deployment simulation via forward kinematics and require additional post-processing to resolve. Developing geometric characteristics for favorable deployment behavior directly from the embedding remains an open problem. Second, patterns that can be uniformly deployed might also be deployed with a different set of hinge angles. Analyzing the degrees of freedom of the deployment space, which could be achieved by sensitivity analysis, is an interesting direction to explore. Third, our framework primarily

focuses on geometric properties in metamaterial design. Establishing connections between these geometric conditions and broader mechanical or material properties, such as stiffness and stability or material thickness [Warisaya et al. 2022; Zaman et al. 2025], would substantially expand the applicability of our work. Finally, our inverse design formulation assumes rigid, planar faces, following prior work [Jiang et al. 2020; Segall et al. 2025]. This assumption restricts the achievable curvature for patterns with large polygonal faces, such as those containing 12-gons (e.g., Fig. 21). Relaxing this constraint to allow controlled bending without stretching could significantly enlarge the design space for shape approximation using new classes of deployable kirigami structures.

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A Proof

Here we prove that, under the uniform deployability assumption, the induced edges in $\mathcal{M}'$ that correspond to a *split cut* in $\mathcal{M}$ remain parallel during deployment at any opening angle. We start with simple cases (Remark A.2 and A.3, based on Remark A.1), and then generalize to the general setting (Remark A.4).

Remark A.1. Consider an interior vertex $v \in \mathcal{M}$. The number of hinge-cut edges directed into v equals the number of hinge-cut edges directed from v .

PROOF. Let $v \in \mathcal{M}$ be an interior vertex, and denote by $\mathcal{F}_v = \{f_i\}_{i=1}^K$ the set of faces incident to v , ordered cyclically. We iteratively merge adjacent faces in $\mathcal{F}_v$ that share the same orientation, stopping when no further merging is possible. This yields a reduced set of faces $\tilde{\mathcal{F}}_v = \{\tilde{f}_j\}_{j=1}^{\tilde{K}}$, in which every pair of adjacent faces has opposite orientations. Because the faces in $\tilde{\mathcal{F}}_v$ alternate in orientation around v , the set $\tilde{\mathcal{F}}_v$ is two-colorable, and therefore $\tilde{K}$ must be even. Note that this merging operation does not eliminate any hinge-cut edges around v ; instead, it only eliminates the split-cut edges, namely those shared by two faces with the same orientation. After this merging, all edges incident to v from $\tilde{\mathcal{F}}_v$ are exactly the hinge-cut edges incident to v from $\mathcal{F}_v$. When these edges are traversed in cyclic order around v , their induced directions alternate between pointing into v and pointing out of v , since any neighboring hinge-cut edges belong to the same face in $\tilde{\mathcal{F}}_v$. Since $\tilde{K}$ is even, the number of hinge-cut edges directed into v equals the number directed out of v . See Fig. 27 for illustrations. $\square$

Remark A.2. Let $v \in \mathcal{M}$ be a vertex incident to *exactly one* split-cut edge $e \in \mathcal{M}$. Let $e', e'' \in \mathcal{M}'$ denote the two edges corresponding to e in the resulting kirigami $\mathcal{M}'$ after cutting. If $\mathcal{M}'$ is uniformly deployable, then e' and e'' remain *parallel* for all deployment angles.

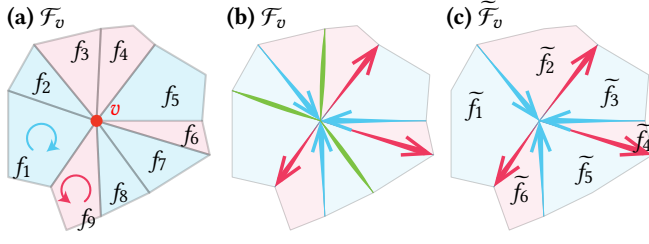


Fig. 27. **Notations for Remark A.1.** (a) The faces $\mathcal{F}_v$ incident to an interior vertex v ; faces assigned clockwise (counterclockwise) orientation are colored blue (pink). (b) The hinge-cut edges (highlighted in blue or red arrows) and split-cut edges (shown in green) around v , as determined by the face orientations. (c) Neighboring faces with the same orientation are merged, yielding a reduced faces set $\tilde{\mathcal{F}}_v$ that is two-colorable. The hinge-cut edges derived from $\tilde{\mathcal{F}}_v$ are identical to those derived from $\mathcal{F}_v$.

PROOF. Let $v \in \mathcal{M}$ be incident to exactly one split-cut edge e . By Remark A.1, the hinge-cut edges occur in equal numbers directed into and out of v , say k of each. Thus, the valency of v is $2k + 1$. Denote the faces incident to v by $f_1, \dots, f_{2k+1}$ in cyclic order, and let α_i denote the angle of face f_i at v . Since $\mathcal{M}$ is planar and gap-free,

$$\sum_{i=1}^{2k+1} \alpha_i = 2\pi. \quad (15)$$

Vertex duplication polygon. After cutting, the split-cut edge and the k hinge-cut edges directed into v each produce a duplicated copy of v . Connecting these $k + 1$ copies cyclically forms a polygon $\mathcal{K}$ with $k + 1$ vertices from $\mathcal{M}'$. Let $\beta_1, \dots, \beta_{k+1}$ denote the interior angles of $\mathcal{K}$. Since $\mathcal{M}'$ is assumed to be deployable, the hole $\mathcal{K}$ remains planar. Therefore we have:

$$\sum_{i=1}^{k+1} \beta_i = (k - 1)\pi. \quad (16)$$

Hinge-cut angle contributions: Each hinge-cut edge forms an isosceles triangle during deployment. See the highlighted purple triangle in Fig. 28 (b). If θ is the opening angle of a hinge and ϕ the base angle, then

$$\theta + 2\phi = \pi. \quad (17)$$

Each hinge-cut edge directed into v contributes two angles of size ϕ , and each hinge-cut edge directed from v contributes one angle of size θ . Thus, the total contribution of the hinge-cut edges to the angle sum around the duplicated vertices is $k(2\phi + \theta) = k\pi$.

Split-cut edge contribution: The unique split-cut edge produces two angles, denoted (x, y) , incident to the two copies of v . These angles determine the relative orientation of the duplicated edges e' and e'' . See Fig. 28 (c).

Angle sum around duplicated copies of v in $\mathcal{M}'$: Since $\mathcal{M}'$ is deployable, the angles incident to each copy of v sum to 2π . Summing over all $k + 1$ copies yields

$$\sum_{i=1}^{2k+1} \alpha_i + \sum_{i=1}^{k+1} \beta_i + k(2\pi + \theta) + (x + y) = 2(k + 1)\pi. \quad (18)$$

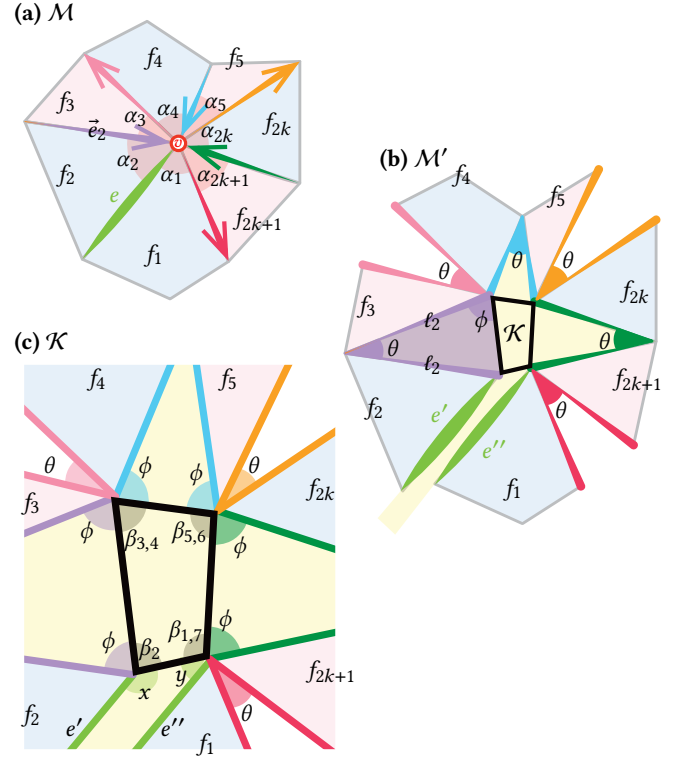


Fig. 28. **Notations for Remark A.2.** (a) α_i denotes the angle of face f_i at the vertex v , and ℓ_i denotes the length of the edge shared by the face f_i and f_{i+1} . (b) Since $\mathcal{M}'$ is assumed to be uniformly deployable, all hinge-cut edges open by the same angle θ . (c) $\mathcal{K}$: the $(k + 1)$ -gon in $\mathcal{M}'$ (here $k = 3$) formed by connecting the duplicated copies of vertex v . $\beta_{i,i+1}$ denotes the interior angle of $\mathcal{K}$ at the vertex incident to the faces f_i and f_j . Note the interior angle β_2 is adjacent only to face f_2 , since the duplicates (e', e'') of the split-cut edge $e = f_1 \cap f_2$ are completely separated after cutting.

See Fig. 28 (c) for an illustration of the example with $k = 3$. Substituting (15), (17) and (16) gives

$$x + y = \pi, \quad (19)$$

which is equivalent to $e' \parallel e''$.

□

Remark A.3. Let $v \in \mathcal{M}$ be an interior vertex incident to multiple split-cut edges, denoted by $\mathcal{E}_{split}^v$. Fix an edge $e_i \in \mathcal{E}_{split}^v$. If the two duplicated copies of every edge $e \in \mathcal{E}_{split}^v \setminus \{e_i\}$ are parallel during deployment, then the duplicated copies of e_i are also parallel.

PROOF. The proof is a direct generalization of the proof of Remark A.2, extending the same angle-sum argument to a vertex incident to multiple split-cut edges.

Let $v \in \mathcal{M}$ be an interior vertex incident to k_1 split-cut edges and k_2 hinge-cut edges directed into v . By Remark A.1, there are k_2 hinge-cut edges directed from v , so the valency of v is $k_1 + 2k_2$.

Vertex duplication polygon: After cutting, each split-cut edge and each hinge-cut edge directed into v produces a duplicated copy of v . These $k_1 + k_2$ copies are connected cyclically to form a polygon $\mathcal{K}$

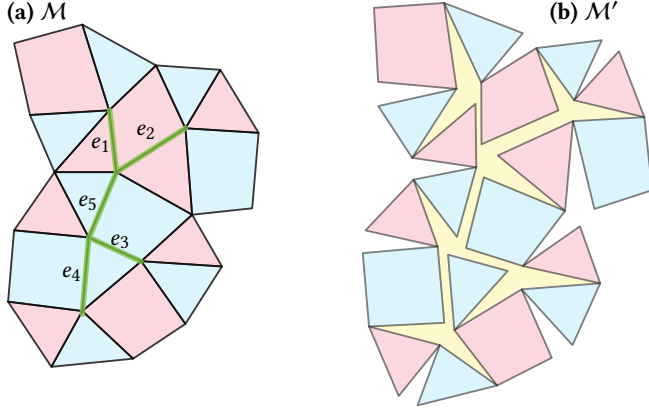


Fig. 29. (a): A hole preimage consists of 5 split-cut edges. The duplicates of e_1, e_2, e_3, e_4 remain parallel after cutting according to Remark A.2. Then, the duplicates of e_5 remain parallel after cutting according to Remark A.3. (b): The resulting kirigami structure with the corresponding hole highlighted.

in $\mathcal{M}'$, generalizing the polygon in Remark A.2. Let $\beta_1, \dots, \beta_{k_1+k_2}$ denote the interior angles of $\mathcal{K}$. Since $\mathcal{K}$ remains planar,

$$\sum_{i=1}^{k_1+k_2} \beta_i = (k_1 + k_2 - 2)\pi. \quad (20)$$

Angle sum around v in $\mathcal{M}'$: Let $\alpha_1, \dots, \alpha_{k_1+2k_2}$ denote the angles of the faces incident to v in $\mathcal{M}$, which satisfy

$$\sum_{i=1}^{k_1+2k_2} \alpha_i = 2\pi. \quad (21)$$

Similar to the discussions in the proof of Remark A.2, each hinge-cut edge directed into v contributes two angles of size ϕ , each hinge-cut edge directed from v contributes one angle of size θ , and $\theta + 2\phi = \pi$. Hence, the total hinge contribution is $k_2(\theta + 2\phi) = k_2\pi$. Each split-cut edge e_j produces two angles (x_j, y_j) .

Angle sum around duplicated copies of v in $\mathcal{M}'$: Since $\mathcal{M}'$ is deployable, angles around each copy of v sum to 2π . Summing over all $k_1 + k_2$ copies gives

$$\sum_{i=1}^{k_1+2k_2} \alpha_i + \sum_{i=1}^{k_1+k_2} \beta_i + k_2\pi + \sum_{j=1}^{k_1} (x_j + y_j) = 2(k_1 + k_2)\pi. \quad (22)$$

Substituting the previous sums yields

$$\sum_{j=1}^{k_1} (x_j + y_j) = k_1\pi. \quad (23)$$

By assumption, $(x_j + y_j) = \pi$ for all $j \neq i$. Therefore,

$$x_i + y_i = k_1\pi - \sum_{j \neq i} (x_j + y_j) = \pi, \quad (24)$$

which shows that the duplicated copies of e_i are parallel. $\square$

Remark A.4. Let $e \in \mathcal{M}$ be a split-cut edge, and let $e', e'' \in \mathcal{M}'$ denote the two edges corresponding to e in the resulting kirigami structure $\mathcal{M}'$ after cutting. If $\mathcal{M}'$ is uniformly deployable, then throughout the deployment process, the edges e' and e'' remain parallel for all deployment angle.

PROOF. Since the holes in $\mathcal{M}'$ are disjoint, it suffices to consider a single hole and show that, for this hole, the duplicated copies of all its split-cut edges remain parallel throughout deployment.

Fix a split-cut edge $e \in \mathcal{M}$, and let $\tilde{C}_e = C_e \cap \mathcal{E}_{split}$ denote the split-cut edges in its associated hole preimage C_e . By construction, the edges in $\tilde{C}_e$ together with their endpoints form a connected *acyclic* graph, hence a tree. Otherwise, the presence of a cycle would completely separate the enclosed faces after cutting, which violates the principles discussed in Appendix 4.5. We prove the claim by induction on the number of edges in $\tilde{C}_e$. If $\tilde{C}_e$ consists of a single edge, the result follows from Remark A.2. Otherwise, consider the tree $\tilde{C}_e$ layer by layer. All split-cut edges incident to leaf vertices have parallel duplicated copies by Remark A.2. Now consider any split-cut edge whose endpoints are incident only to such leaf edges. By Remark A.3, its duplicated copies are also parallel. Removing the leaf edges produces a smaller tree with the same property. Iterating this peeling process from the boundary inward shows that every split-cut edge in $\tilde{C}_e$ has parallel duplicated copies. $\square$